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Note

A Characterization of Best Simultaneous Approximations

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1. INTRODUCTION

Let X be a compact Hausdorff space and Y a normed linear space with norm $\|\cdot\|$. Let $C(X, Y)$ denote the set of all continuous functions from X to Y. Suppose that functions $F_1, ..., F_l$ in $C(X, Y)$ are given. We want to approximate these functions simultaneously by functions in S, an n-dimensional subspace of $C(X, Y)$, in the sense of Chebyshev. That is, the problem is to find a function $f \in S$ which minimizes

$$\max_{1 \leqslant j \leqslant l} \max_{x \in X} \|F_j(x) - f(x)\| \tag{1}$$

over the set S. If such a function f^* in S exists, we call it a best simultaneous approximation for $F_1, ..., F_l$. The purpose of this note is to derive a necessary and sufficient condition for a function to be a best simultaneous approximation.

To this end, first remark that

$$\max_{1 \leqslant j \leqslant l} \|F_j(x) - f(x)\| = \max_{a \in A} \left\| \sum_{j=1}^{l} \alpha_j F_j(x) - f(x) \right\|, \tag{2}$$

where the set A is defined by

$$A = \left\{ a = (\alpha_1, ..., \alpha_l) \,\middle|\, \sum_{j=1}^{l} \alpha_j = 1, \alpha_j \geqslant 0 \; (1 \leqslant j \leqslant l) \right\}.$$

This follows from the expression

$$\sum_{j=1}^{l} \alpha_j F_j(x) - f(x) = \sum_{j=1}^{l} \alpha_j (F_j(x) - f(x))$$

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and the inequalities

$$\max_{1 \leqslant j \leqslant l} \|y_j\| \leqslant \max_{a \in A} \left\| \sum_{j=1}^{l} \alpha_j y_j \right\| \leqslant \max_{a \in A} \sum_{j=1}^{l} \alpha_j \|y_j\| \leqslant \max_{1 \leqslant j \leqslant l} \|y_j\|,$$

where $y_1, ..., y_l$ are arbitrary elements of the normed linear space. Then (1) can be expressed as

$$\max_{a \in A} \max_{x \in X} \left\| \sum_{j=1}^{l} \alpha_j F_j(x) - f(x) \right\|.$$

Furthermore, if we regard the set A as the set of l-dimensional row vectors, and denote by $F(x)$ the column vector $(F_1(x), ..., F_l(x))^t$, $\sum_{j=1}^{l} \alpha_j F_j(x)$ can be denoted by the inner product $aF(x)$ of two vectors a and $F(x)$. Thus the problem takes on the expression

$$\text{minimize} \max_{(a,x) \in A \times X} \|aF(x) - f(x)\| \text{ over the set } S. \tag{3}$$

Note that the set $A \times X$ is compact.

2. Best Simultaneous Approximation

For $f \in C(X, Y)$, we define the uniform norm of f by

$$|||f||| = \max_{x \in X} \|f(x)\|,$$

and endow the linear space $C(X, Y)$ with the uniform topology. It is easily checked that $\|aF(x) - f(x)\|$ is a jointly continuous function of the three variables a, x, f and convex in f, that is,

$$\|aF(x) - (\theta f + (1-\theta) g)(x)\| \leqslant \theta \|aF(x) - f(x)\| + (1-\theta) \|aF(x) - g(x)\|$$

for all $f, g \in S$ and θ, $0 \leqslant \theta \leqslant 1$. We state the main result, which is an analog of Theorem 4.1 of [1] and also a generalization of it.

THEOREM. *A function $f^* \in S$ is a best simultaneous approximation if and only if there exist $\lambda_1^*, ..., \lambda_k^* > 0$, $\sum_{i=1}^{k} \lambda_i^* = 1$, k distinct elements $x_1^*, ..., x_k^* \in X$, and k vectors $a_1^*, ..., a_k^* \in A$, where $1 \leqslant k \leqslant n+1$, such that*

$$\text{(i)} \qquad \|a_i^* F(x_i^*) - f^*(x_i^*)\| = \max_{1 \leqslant j \leqslant l} \|F_j(x_i^*) - f^*(x_i^*)\|$$

$$= \max_{1 \leqslant j \leqslant l} |||F_j - f^*|||, \qquad i = 1, ..., k;$$

$$\text{(ii)} \quad \sum_{i=1}^{k} \lambda_i^* \|a_i^* F(x_i^*) - f(x_i^*)\| \geqslant \sum_{i=1}^{k} \lambda_i^* \|a_i^* F(x_i^*) - f^*(x_i^*)\|$$

for all $f \in S$.

Proof. By the preceding remarks concerning the function $\|aF(x) - f(x)\|$, we can apply Corollary 3.3 of [1] to problem (3). The rest of the proof is similar to that of Theorem 4.1 of [1]. The only difference is in the additional existence of k vectors $a_1^*, ..., a_k^* \in A$ and, in accordance with it, relation (2) is used in the derivation of the necessary condition (in particular, condition (i)).

Assume that the set X contains more than n points and S is an n-dimensional Haar subspace of $C(X, Y)$ (see [1]). Then we conclude that $k = n + 1$, unless $F_1 = \cdots = F_l \in S$. It can be shown by the method employed in the proof of Theorem 5.1 of [1].

Reference

1. S. Tanimoto, Uniform approximation and a generalized minimax theorem, *J. Approx. Theory* **45** (1985), 1–10.